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APPLICATION OF A MARKOV CHAIN TO A GENERALIZATION OF AN ALGEBRAIC IDENTITY

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Department of the Army Project No. 503-06-002
Ordnance Management Structure Code No. 5010.11.812
BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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ABSTRACT

An identity $$\frac{x_1^n - x_2^n}{x_1 - x_2} = \sum_{i_1 + i_2 = n-1} x_1^{i_1} x_2^{i_2}$$

is here generalized to an arbitrary number of variables $x_1, \ldots, x_m$. The proof of the generalized identity

$$\sum_{i_1+i_2+\ldots+i_m = n-m+1} x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m} = \frac{x_1^n}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)} + \frac{x_2^n}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)} + \ldots + \frac{x_m^n}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})}$$

is obtained by computing transition probabilities in a Markov Chain.

Assume that $0 < x_i < 1$, $i = 1, 2, \ldots m$, $x_i \neq x_j$ for $i \neq j$, and let $y_i = 1 - x_i$. Consider a Markov Chain with the matrix of transition probabilities:

$$\begin{pmatrix} x_1 & y_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & y_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & y_3 & 0 & \ldots & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & \ldots & x_m \end{pmatrix}$$

If $n \geq r$ then a transition of the system from the state k to a state $k + r$ in n steps can occur in any of the set of mutually exclusive ways, each corresponding to a distinct sequence of integers $t_1, t_2, \ldots, t_r$, satisfying the inequalities $1 \leq t_1 < t_2 < \ldots < t_r \leq n$. In each of these cases the system passes from the state $k + h$ to the state $k + h + 1$ at the step number t_{h-1} . The probability of such an event is $y_k y_{k+1} \ldots y_{k+r-1} x_k^{t_1 - 1} x_{k+1}^{t_2 - t_1 - 1} \ldots x_{k+r}^{n - t_r}$. Here the sum of the exponents of x_{k+i}'s is $t_1 - 1 + t_2 - t_1 - 1 + \ldots + n - t_r = n - r$. Therefore the probability $p_{kr}^{(n)}$ of transition in n steps from the state k to the state $k + r$ is given by

$$(1) \qquad p_{kr}^{(n)} = y_k y_{k+1} \ldots y_{k+r-1} \sum_{i_1 + \ldots + i_{r+1} = n - r} x_k^{i_1} x_{k+1}^{i_2} \ldots x_{k+r}^{i_{r+1}}$$

On the other hand, $p_{kr}^{(n)}$ can be obtained by the method described in

[1]. Let D(s) be the determinant of the system

$$
(2)\qquad \begin{aligned} \xi_j &= s(x_j\,\xi_j + y_j\,\xi_{j+1}) , \quad j = 1, 2, \ldots, m-1 , \\ \xi_m &= s\,x_m\,\xi_m , \end{aligned}
$$

and let $(\xi_1^{(t)}, \ldots, \xi_m^{(t)})$ be a non-trivial solution of (2) corresponding to the root s_t of D(s). Similarly, let $(\eta^{(t)}, \ldots, \eta_m^{(t)})$ be a non-trivial solution of

$$
(3)\qquad \begin{aligned} \eta_1 &= s\,x_1\eta_1 , \\ \eta_i &= s(y_{i-1}\eta_{i-1} + x_i\,\eta_i) , \end{aligned}
$$

corresponding to the same root s_t of D(s).

Let

$$
(4)\qquad c_t = \frac{1}{\sum_{\nu=1}^{m} \xi_\nu^{(t)}\,\eta_\nu^{(t)}} ,
$$

and

$$
(5)\qquad \zeta_{ji}^{(t)} = c_t\,\xi_j^{(t)}\,\eta_i^{(t)} .
$$

Then

$$
(6)\qquad p_{kr}^{(n)} = \frac{\zeta_{k\,k+r}^{(1)}}{s_1^n} + \frac{\zeta_{k\,k+r}^{(2)}}{s_2^n} + \ldots + \frac{\zeta_{k\,k+r}^{(m)}}{s_m^n}
$$

It follows from (2) that non-trivial solutions $(\xi_1^{(t)}, \ldots, \xi_m^{(t)})$ can be obtained for the following values of s: $s_1 = \frac{1}{x_m}$, $s_2 = \frac{1}{x_{m-1}}$, $\ldots$, $s_m = \frac{1}{x_1}$. By substituting $s_t = \frac{1}{x_{m-t+1}}$ in (2) the following expressions are obtained:

(7) $$\xi_m^{(t)} = \xi_{m-1}^{(t)} = \ldots. = \xi_{m-t+2}^{(t)} = 0 \quad ,$$

(8) $$\xi_{m-t+1}^{(t)} = 1 \quad ,$$

The substitution of (8) in the first equation of (2) yields

(9) $$\xi_{m-t} = \frac{y_{m-t}}{x_{m-t+1} - x_{m-t}}$$

Suppose, by induction, that for $j \leqslant m - t$

(10) $$\xi_j^{(t)} = \prod_{p=0}^{m-t-j} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}}$$

The substitution of (10) in (2) yields

$$\xi_{j-1}^{(t)} = \prod_{p=0}^{m-t-j+1} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}}$$

Thus, in view of (9), (10) holds for every $j \leqslant m - t$.

Similarly, it follows from (3) that

$$\eta_i^{(t)} = 0 \ , \quad \text{for } i < m - t + 1 \ ;$$

(11) $$\eta_{m-t+1}^{(t)} = 1 \ ;$$

$$\eta_i^{(t)} = \prod_{p=0}^{i-m+t-2} \frac{y_{i-p-1}}{x_{m-t+1} - x_{j-p}} \ , \quad \text{for } i > m - t + 1$$

Therefore it follows from (4), (7), (8), and (11) that

(12) $$c_t = 1 \ , \quad t = 1, 2, \ldots, m.$$

Now (5), (7), (8), (10), and (12) yield

$$\zeta_{ji}^{(t)} = 0\ , \quad \text{if } i < m - t + 1 \quad \text{or} \quad j > m - t + 1 \ ;$$

$$\zeta_{m-t+1\ m-t+1}^{(t)} = 1 \ ;$$

(12)

$$\zeta_{m-t+1\ i}^{(t)} = \prod_{p=0}^{m-t-i-2} \frac{y_{i+p-1}}{x_{m-t+1} - x_{i+p}}\ , \quad \text{if } i > m - t + 1 \ ;$$

$$\zeta_{j\ m-t+1}^{(t)} = \prod_{p=0}^{m-t-j} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}}\ , \quad \text{if } j < m - t + 1$$

$$\zeta_{ij}^{(t)} = \prod_{p=0}^{m-t-j} \frac{y_{j+p}}{x_{m-t+1} - x_{j+p}} \prod_{q=0}^{i-m+t-2} \frac{y_{i-q-1}}{x_{m-t+1} - x_{j-q}}\ ,$$

$$\text{if } i > m - t + 1 \text{ and } j < m - t + 1 \ .$$

Therefore by (6)

(13) $$p_{1,m}^{(n)} = y_1\, y_2 \ldots y_{m-1} \left[\frac{x_m^n}{(x_m - x_{m-1})(x_m - x_{m-2}) \ldots (x_m - x_1)} + \right.$$

$$+ \frac{x_{m-1}^n}{(x_{m-1} - x_m)(x_{m-1} - x_{m-2}) \ldots (x_{m-1} - x_1)} + \ldots + $$

$$\left. + \frac{x_1^m}{(x_1 - x_m)(x_1 - x_{m-1}) \ \ldots \ (x_1 - x_2)} \right]$$

Let $P_k\,(x_1,\ x_2,\ \ldots x_m) = \sum_{i_1+i_2+\ldots+i_m=k} x_1^{i_1}\, x_2^{i_2} \ldots x_m^{i_m}$

Then by (1) and (13)

$$(14) \qquad P_{n-m+1}(x_1, x_2, \ldots, x_m) = \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_m \\ x_1^2 & x_2^2 & \cdots & x_m^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{m-2} & x_2^{m-2} & \cdots & x_m^{m-2} \\ x_1^n & x_2^n & \cdots & x_m^n \end{vmatrix}}{V(x_1, x_2, \ldots, x_m)} ,$$

where V is a Vandermonde's determinant. Now the numerator N of the right hand side member of (14) is a polynomial in $x_1, \ldots, x_m$ and it vanishes whenever $x_i = x_j$ for $i \neq j$. Since $V = (x_1 - x_2) \ldots (x_1 - x_m)(x_2 - x_3) \ldots (x_{m-1} - x_m)$, it follows that N is divisible by V. Hence (14) is an equality of two polynomials which is valid for every x_i such that $0 < x_i < 1$. Hence (14) is an identity whenever N/V is defined, i.e. whenever $x_i \neq x_j$ for $i \neq j$. This identity can be written in the form

$$(15) \qquad P_{n-m+1} = \frac{x_1^n}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)} + \frac{x_2^n}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)} + \ldots + \frac{x_m^n}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})} .$$

For $m = 2$ (15) reduces to

$$x_1^{n-1} + x_1^{n-2}x_2 + x_1^{n-3}x_2^2 + \ldots + x_2^{n-1} = \frac{x_1^n}{x_1-x_2} + \frac{x_2^n}{x_2-x_1} .$$

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REFERENCE

1 W. Feller, *An Introduction to Probability Theory and its Applications*, John Wiley and Sons, Inc., Chpt. 16.

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